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LETTER TO THE EDITOR

Exact solution for a model on directed lattices

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Abstract. We define a model for information propagation through the oriented bonds of a square lattice. The source of information is the origin, and the propagation takes place in only one quadrant, along the diagonal t (time) direction. The information reaches increasing-t layers of lattice points (r, t) according to a probability distribution $P(x-x_c, r, t)$ dependent only on the previous $P(x-x_c, r, t-1)$, where x is the concentration of present bonds. The model shows two distinct behaviours for large values of t: the information will disappear if $x<x_c$, or survive forever if $x>x_c$. Taking advantage of the Markovian behaviour and assuming that P is homogeneous, we get the values $x_c=\frac{1}{2}$ for the critical concentration, $\nu=1$ for the t-correlation length critical exponent, $\theta=2$ for the t/r crossover exponent, and $\beta=\frac{1}{2}$ for the order parameter exponent. Along directions other than t, from the same origin, we get $\nu_1=1$. The homogeneity assumption is supported by numerical calculations of the time evolution. This evolution is a deterministic cellular automaton, each cell r retaining a real (instead of discrete) value for P. The similarities and differences between the present model and directed bond percolation are also discussed.

The study of phase transitions in directed lattices is very interesting for at least two reasons. From a practical point of view, the one-sense character of these problems allows some simplifications like the reduction of the number of configurations that appear in real space [1] and phenomenological [2, 3] renormalisation group treatments, or in Monte Carlo simulations [4]. From another point of view, the directionality introduces an anisotropy between the 'time' and the transverse directions [5]. New universality classes and scaling relations [6] arise from this anisotropy. Another interesting feature of those directed problems is their relation with cellular automata [7]. There are also some exactly known results for directed problems [3, 8]. The models usually treated in directed lattices are percolation and lattice animals.

In this letter we define a particular model for information propagation along a directed lattice. Consider the directed square lattice (see figure 1), the extension to other lattices or other dimensions is straightforward. A fraction x of the bonds are present according to a random distribution. Each bond can carry information only in the sense of the lattice orientation. The information propagation starts at the origin and flows through the present bonds. We define a t (time) axis along the principal diagonal, and a r axis perpendicular to it (see figure 1). First we consider the probability distribution $P(x-x_c, r, t)$ that a given point (r, t) will be reached by the information that is passed to it from line $t-1$ to line t. For $t=1$, $P(x-x_c, r, 1)$ is clearly equal to x for $r=1$ and $r=-1$, the distribution being two delta functions. To obtain $P(x-x_c, r, t)$ from the already known $P(x-x_c, r, t-1)$, we follow the procedure below.

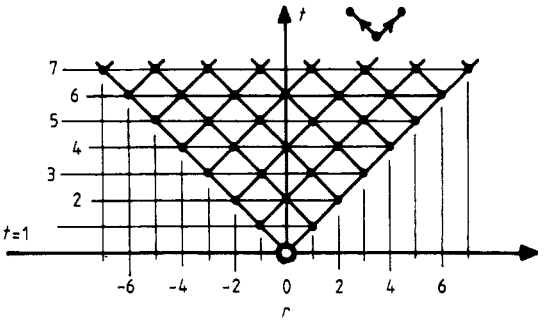

Figure 1. Directed square lattice on which the information generated at the origin diffuses through the bonds following their orientation.

(i) We choose a random configuration for the line $t-1$, each point retaining or not the information according to the known distribution $P(x-x_c, r, t-1)$.

(ii) We choose a random configuration for the bonds linking lines $t-1$ and t, each bond present (or not) with probability x (or $1-x$).

(iii) We increment by one a counter $c(r)$ (initially zeroed) for each point (r, t) which is reached by the information.

(iv) We repeat a, b and c, N times, N being large.

(v) We define $P(x-x_c, r, t)$ as the ratio $c(r)/N$.

We can think about this model as a learning process in which the probability that a student (point in line t) succeeds in retaining the information is proportional to the number of times his teachers (neighbour points in line $t-1$) have given it to him. The above definition of our model yields the recursion relation

$$1-P(r,t)=[1-xP(r-1,t-1)][1-xP(r+1,t-1)]. \tag{1}$$

The time evolution of $P(x-x_c, r, t)$ can be viewed as a deterministic cellular automaton [9] defined by equation (1), each cell r retaining the current value of P. This automaton, instead of spanning a particular configuration of the lattice like the one defined by Domany and Kinzel [7] for the directed percolation problem, determines the probability distribution itself. Depending on the value of x, the information can vanish for large values of t, or can survive forever. We can see this phase transition by focusing on the behaviour of the cell $r=0$ (for which P is obviously maximal). If $P(r=0)$ converges to a finite value P^* for large t, the information succeeds in surviving forever, while it vanishes if $P^*=0$. From equation (1) we can determine the possible values of P^*:

$$1-P^*=[1-xP^*]^2. \tag{2}$$

Equation (2) has two roots $P_1^*=(2x-1)/x^2$ and $P_2^*=0$. The attractor of map (1) (along the t axis) is $P_1^*>0$ for $x>\frac{1}{2}$, and becomes $P_2^*=0$ for $x<\frac{1}{2}$, as can be seen by linearising (1) around P^*. So, the critical concentration for our model is $x_c=\frac{1}{2}$.

Now, we assume that P is a generalised homogeneous function of its arguments, as in equation (3), where $\varepsilon=x-x_c$ and Ω is an arbitrary scaling factor:

$$P(\Omega^{-1/\nu}\varepsilon, \Omega^{1/\theta}r, \Omega t)\Omega^{-a}P(\varepsilon, r, t) \qquad \{a=1/\nu\}. \tag{3}$$

The value $a=1/\nu$ was obtained noting that $P(\varepsilon, 0, \infty)=P_1^*$ is linear in $\varepsilon>0$ near the critical point.

Equation (3) together with its second r derivative, equation (1) and the symmetry $P(-r) = P(r)$ gives us the asymptotic behaviour of P and d^2P/dr^2, for $\varepsilon = 0$ and $r = 0$, as shown in equations (4) and (5):

$$P(0, 0, t) = Ct^{-1/\nu} \qquad C > 0 \tag{4}$$

$$P(0, 0, t) - P(0, 2, t) = Kt^{-1/\nu - 2/\theta} \qquad K > 0. \tag{5}$$

Figure 2 shows log-log plots obtained by iterating the cellular automaton (1) numerically. From those plots we can determine the values of the critical exponents ν and θ, as well as those of the constants C and K. We obtain the numerical values $\nu = 1.00$, $\theta = 2.00$, $C = 2.7$ and $K = 3.4$ with the precision indicated by the displayed digits. Now we substitute the asymptotic equations (4) and (5) into (1), and obtain relation (6) retaining only the leading terms

$$(4/\nu)t^{-1} = Ct^{-1/\nu} + (K/C)t^{-2/\theta}. \tag{6}$$

From (6) we can have only three possibilities:

(i) $\nu = 1$ $\theta = 2$; $C + (K/C) = 4$

(ii) $\nu = 1$ $\theta < 2$; $C = 4$

(iii) $\nu < 1$ $\theta = 2$; $(K/C) = 4/\nu$.

Possibilities (ii) and (iii) are immediately discarded by our previous numerical results concerning the constants C and K. We can thus conclude that possibility (1) is the correct one, and gives the exact values for the critical exponents ν and θ, providing the scaling hypothesis (3) is correct. Figure 3 shows a series of data collapsing plots that support this hypothesis.

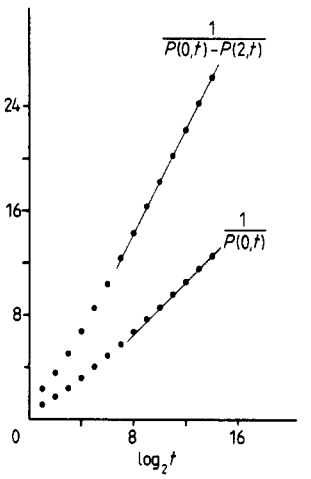

Figure 2. Log-log plots for P and its second r derivative at the critical concentration for $r = 0$.

Like in directed percolation, our model presents different correlation length critical exponents along the axes t and r ($\theta > 1$ is the ratio between these two exponents). At the critical concentration, the information diffuses inside a cone from the origin, and the width $\langle r \rangle$ of this cone at line t scales non-linearly with t, i.e. $\langle r \rangle \sim t^{1/\theta}$. Thus, at the critical concentration, the information diffuses asymptotically *only* in the t direction, the *angular* opening of the cone being zero. Although the number of sites reached at

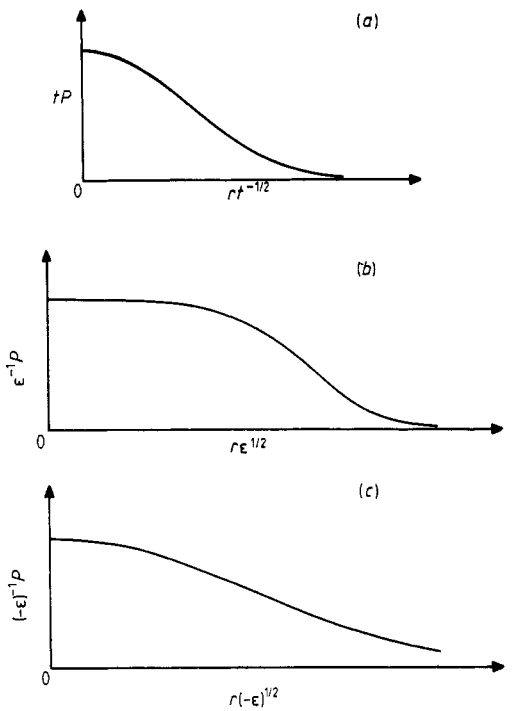

Figure 3. Data collapsing plots to support the scaling hypothesis (3). (*a*) $\varepsilon = 0$, $t = 8192$ and 16 384. (*b*) $(\varepsilon, t) = (+0.0005, 8192)$ and $(+0.000\,25, 16\,384)$. (*c*) $(\varepsilon, t) = (-0.000\,01, 8192)$ and $(-0.000\,005, 16\,384)$.

line t grows to infinity, they represent a null fraction of sites in that line, at the critical concentration. Above the critical concentration, however, the *angular* opening 2Ψ of the cone becomes an increasing function of the concentration, giving a measure for the fraction of sites reached (the order parameter).

For $x > x_c$ the model also presents another transition, from the point of view of an observer located far from the origin along a direction $\Psi > 0$. We can define a Ψ-dependent critical concentration $\varepsilon(\Psi) > 0$ below which that observer is not reached by the information. The corresponding correlation length critical exponent ν_1 does not depend on $\Psi > 0$. By arguing that different concentrations x and y for bonds along the two lattice directions (maintaining $\Psi = 0$) is equivalent to $\Psi > 0$ (maintaining $x = y$), Domany and Kinzel [3] are able to show that $\nu_1 = 2$ for directed percolation, by an exact calculation performed for $y = 1$, and also by a phenomenological RG. RG flows in x-y space confirm the idea that correlations decay with distinct exponents ν for $\Psi = 0$ and ν_1 for $\Psi > 0$ (see de Oliveira, Kamphorst Leal da Silva and Droz [1]). The relation between the exponents ν, θ and ν_1 is not yet understood for percolation, but we can investigate this point for our model. The distribution $P(\varepsilon, r, t)$ has another scaling property for $\varepsilon > 0$ besides the one already shown in figure 3(*b*): for a given value of $x > x_c$, $P(r, t)$ propagates in r space like a soliton with a constant speed Ψ, maintaining the same wavefront form (see figure 4(*a*)). For increasing values of t, the distribution P is not modified for an observer attached to the point $r_0(t) = \Psi t$ where $P = P^*/2$. This is very well established from our numerical results exemplified in figure 4(*a*). In the neighbourhood of r_0, where the wavefront can be approximated to be straight, we can understand this behaviour: due to the local character of equation (1),

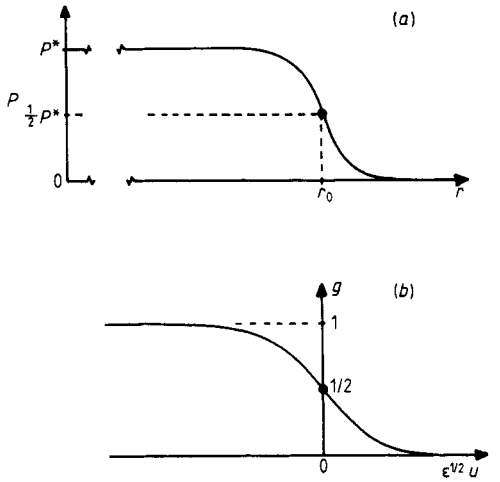

Figure 4. The distribution function $P(\varepsilon, r, t)$ for $\varepsilon > 0$. (*a*) Original form for $x = 0.51$, and (*b*) after changing the origin of the r axis to $r_0(t) = \Psi t$, and taking the scaling (7), for $t = 4096$, 8192, 16 384 and $\varepsilon = 0.01$, 0.004, 0.002.

the value of P will undergo the same shift for any point inside this neighbourhood. The scaling hypothesis (3) can be used to define another universal function g in equation (7), shown in figure 4(*b*):

$$P(\varepsilon, r, t) = P^* g(\varepsilon^{1/2} u) \qquad \{u = r - \Psi t\}. \tag{7}$$

By linearising $P(\varepsilon, r, t-1)$ around $r = \Psi(t-1)$, where we also have $P = P^*/2$, and by applying equation (1) at $r = r_0(t)$, we can get the function $\Psi(x)$ near the critical concentration, shown in equation (8), and the order parameter critical exponent $\beta = \frac{1}{2}$:

$$\varepsilon = D\Psi^2 \qquad \{D \propto g'(0)^2\}. \tag{8}$$

The value of P is asymptotically constant for increasing values of t, along the direction $\Psi(x) > 0$. Along another direction $\Psi(x) + \delta\Psi$, where $\delta\Psi > 0$ is a small deviation, however, P decays exponentially. By exploring the corresponding exponential decaying of $P \propto \exp(\Gamma u)$ for $r \gg r_0$ (at a fixed t), equation (1) gives $\Gamma \propto g'(0)\varepsilon^{1/2}$. Along $\Psi(x) + \delta\Psi$, for increasing values of t, we get $P \propto \exp(t\Gamma\delta\Psi)$. From the derivative of (8) we can finally find the concentration dependence of the inverse correlation length $\Gamma\delta\Psi \sim \delta\varepsilon$, which gives the exact value $\nu_1 = 1$. The exponents ν and ν_1 correspond to distinct critical behaviours and are obtained from distinct scaling features of the distribution function P, and the fact that both have the same value for our model is a particular feature (for directed percolation their values are different).

Besides the similarities, there is a fundamental difference between directed percolation and our model. The equivalent to equation (1) for directed percolation is given in equation (9):

$$P_0(r, t) = x(1-x)P_0(r-1, t-1) + x(1-x)P_0(r+1, t-1) + x^2 P_{-11}(r, t-1). \tag{9}$$

To understand equation (9), let us analyse figure 5. $P_0(r, t)$ is the probability that site (r, t) is reached by the information generated at the origin. We can have three possibilities, one for each term on the right-hand side of equation (9), concerning the presence or absence of the two first bonds linking 0 to 1 and -1. The last term

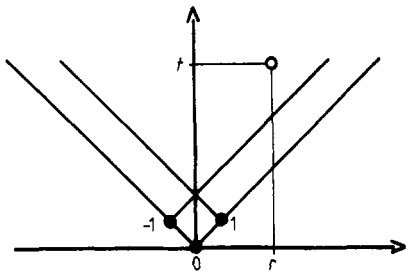

Figure 5. Directed square lattice with the main root and the two secondary roots.

corresponds to the case when both bonds are present, which corresponds to the information being generated *simultaneously* at points 1 and -1. This is an *interference* term similar to the one for the two slits of the classical Young experiment. For directed percolation, unlike our model, this term cannot be split into two, each one depending only on one source of information. Equation (10) gives the difference $\delta P(r, t) = P(r, t) - P_0(r, t)$ between directed percolation and our model:

$$\{\delta P(r,t) - x(1-x)[\delta P(r-1,t-1) + \delta P(r+1,t-1)]\}/x^2$$
$$= [1 - P_{-11}(r,t-1)] - [1 - P(r-1,t-1)][1 - P(r+1,t-1)]. \qquad (10)$$

If we neglect this difference, we can consider our model as a mean field approximation for directed percolation, in the sense that we are substituting the 'two-root term' $1 - P_{-11}$ by an interference-free product of two 'one-root terms'.

In conclusion, we succeeded in obtaining the exact critical concentration and exponents for a model defined on directed lattices. The model has an interesting anisotropy due to the orientation of the lattice, and its features have been understood by analysing the scaling properties of the probability distribution. A possible future extension of this work is the study of the equivalent scaling properties for the directed percolation problem.

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